Recognizable Picture Languages and Polyominoes

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Outline

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1. **Polyominoes**
   - Basic definitions
   - Particular new families of polyominoes
2. **Tiling recognizability**
   - Basic definitions
   - Method to check tiling recognizability
3. **Polyominoes & recognizability**
   - Recognizability of new particular classes of polyominoes
   - Particular new families of polyominoes

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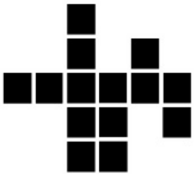
Polyominoes

$\mathcal{P}$



[S.W. Golomb: Polyominoes (1954)]
[M. Gardner: Mathematical Games (1957)]

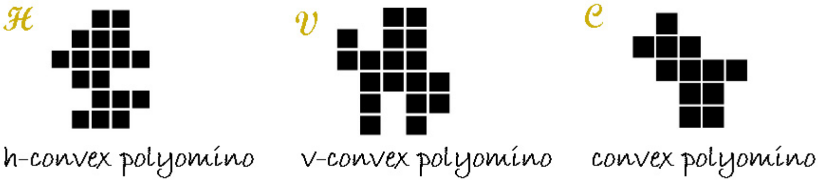
A polyomino is a finite union of elementary cells of the lattice $\mathbb{Z}^2$ defined up to translations (discrete set) in which each cell is connected to each other.

✓ *Two cells of a discrete set S are said to be connected if they can be joined with a path (i.e. sequence of adjacent cells), included in S.*

Polyominoes

- Decidability problems concerning the tiling of the plane using polyominoes [Conway-Lagarias (1990), Beauquier-Nivat(1991)];
- Enumeration problems [Delest-Viennot(1984)];
- Reconstruction of polyominoes from partial informations (for example discrete tomography) [Ryser(1956), Barcucci-Del Lungo et al.(1996), Kuba-Balogh(2002)] ;

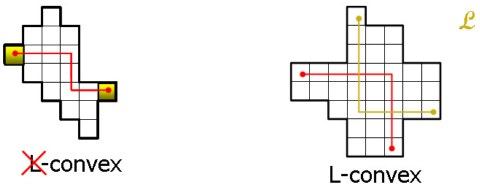
Convex Polyomino ⇢ polyomino in which every row and column is connected



L-convex Polyominoes

Proposition: A polyomino P is convex iff every pair of cells can be connected by a monotone path.

An L-convex polyomino p is a convex polyomino in which every pair of cells is connected by a monotone path, of cells of p, with at most one change of direction. Such a path is called (cause its shape) L-path.



Monotone path: Self-avoiding path that consists of steps in, at most, two directions.

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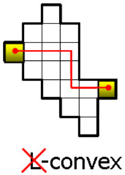

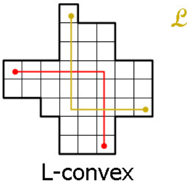

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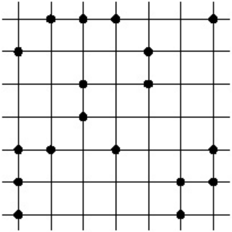
A discrete set and its representations in terms of a binary matrix and a set of cells.



$\longleftrightarrow$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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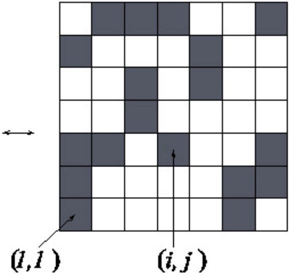
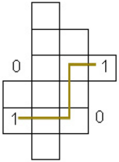
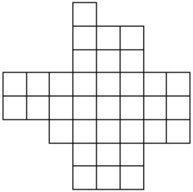


Switching component

Definition: A switching component of a binary matrix is a 2×2 submatrix of either of the following two forms:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$





Lemma: Each L-convex polyomino has no switching component

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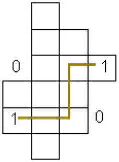

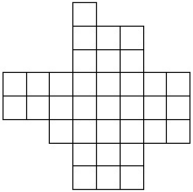

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Switching component

Theorem: A convex discrete set

is L-convex iff it has no switching component

$\mathcal{U}$ $\dashrightarrow$ Family of discrete sets with no switching component

$$\mathcal{L} = \mathcal{C} \cap \mathcal{U}$$

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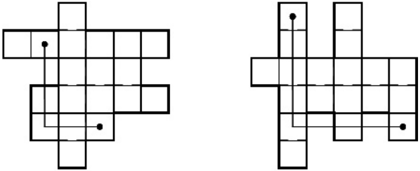

$$\mathcal{L} \subseteq \mathcal{L}_h \subset \mathcal{U} \ , \ \mathcal{L} \subset \mathcal{L}_v \subseteq \mathcal{U},$$

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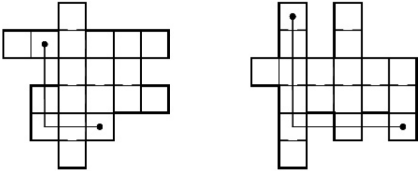

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Picture Languages

A picture is a two dimensional rectangular array of elements in a finite alphabet Σ.

Σ^{**} all pictures over Σ

$p \in \Sigma^{m,n}$ has size (m, n)

$L \subseteq \Sigma^{**}$ ←-- **picture language**

Boundaries of a picture

$\widehat{p} =$

#	#	#	#	#
#	p_{11}	…	p_{1n}	#
#	⋮	⋱	⋮	#
#	p_{m1}	…	p_{mn}	#
#	#	#	#	#

Column concatenation ⇨ pq =

p	q

Recognizable Picture Languages and Tiling System

A picture language L over Σ is called local if there exists a finite set Θ of tiles over $\Sigma \cup \{\sharp\}$ such that $L = \{p \mid p \in \Sigma^{*,*} \text{ and } B_{2,2}(\hat{p}) \subseteq \Theta\}$.

L is recognizable by tiling system(or equivalently *tiling recognizable*) if $L = \pi(L')$ where L' is a local language and π is a mapping from the alphabet Σ of L' to the alphabet Γ of L

projection of a picture $\dashrightarrow$ $p(i,j) = \pi(p'(i,j)) \ \forall i,j$

tile: a square picture of size $(2,2)$

$(\Sigma, \Gamma, \Theta, \pi)$, where $L' = L(\Theta)$, is called tiling system

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Polyominoes as pictures

$\mathcal{H}, \mathcal{V}, \mathcal{C} \dashrightarrow$ *TilingRecognizable*

[*F.DeCarli, A.Frosini, S.Rinaldi, L.Vuillon*]

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How to prove the non-recognizability of a picture language?

Lemma (Matz)

Let $L \subseteq \Sigma^{*,*}$ be tiling recognizable. Let $\{M_n\}_{n \in \mathbb{N}}$ be a sequence of sets $M_n \subseteq \Sigma^{n,+} \times \Sigma^{n,+}$ such that $\forall n$ following relations hold:

$$\forall (p,q) \in M_n \text{ we have } pq \in L \tag{1}$$

$$\forall (p,q) \neq (p',q') \in M_n \text{ we have } \{p'q, p'q\} \nsubseteq L. \tag{2}$$

Then $|\ M_n\ |$ is $2^{O(n)}$.

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Then $|\, M_n \,|$ is $2^{O(n)}$.

Is the language of squares over $\{a, b\}$ that have as many a's as b's an example of non-recognizable picture language for which Matz's Lemma fails to prove the non-recognizability ?

Theorem: (Reinhardt)

The language of picture over $\{a, b\}$, where the number of a's is equal to the number of b's and having a size (m, n) with $m < 2^n$ and $n < 2^m$, is recognizable.

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L-convex ?

Theorem

Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of sets $M_n \subseteq \Sigma^{n,+} \times \Sigma^{n,+}$, with $\Sigma = \{0,1\}$. For all $n \in \mathbb{N}$, if M_n satisfies relations (1) and (2) with respect to the language $\mathcal{L}$ of L-convex polyominoes then $\mid M_n \mid$ is $2^{O(n)}$.

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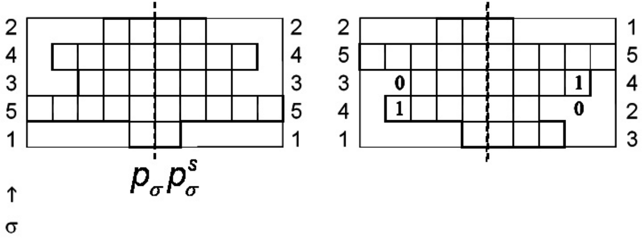
Idea of the proof:



$$M_n = \{(p_\sigma, p_\sigma^s) \mid \sigma \in S_n\} \qquad |M_n| = |S_n| = n!$$

Theorem: $\mathcal{U}$ is not tiling recognizable.

$$\mathcal{L} \subsetneq \mathcal{L}_h \subsetneq \mathcal{U}\,,\ \mathcal{L} \subsetneq \mathcal{L}_v \subsetneq \mathcal{U}$$

Non – recognizable

$$\mathcal{L} = \mathcal{L}_h \cap \mathcal{V} = \mathcal{L}_v \cap \mathcal{H} = \mathcal{C} \cap \mathcal{U}$$

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Recognizable

Non – recognizable

Conjecture: $\mathcal{L}$ is not tiling recognizable.